

# A center vortex representation of the classical SU(2) vacuum

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## Abstract

The classical massless SU(2) field theory has an infinite number of gauge equivalent representations of the vacuum. We show that among these there exists a non-perturbative center vortex representation with some similarity to the quantum vacuum of the same theory. This classical SU(2) vacuum consists of a lattice of center vortex pairs combining to triviality. However, this triviality can be broken by perturbations, for example by adding a mass term, or considering the electroweak theory where the Higgs field does the breaking, or by quantum fluctuations like in QCD.

In non-Abelian field theories the physical vacuum can be considerably complicated. For example, in the quantum QCD vacuum a condensate of center vortices is expected. Recent discussions of the SU(2) case can be found in ref. [1] and [2]. In the following we shall show that something similar is the case even for the classical vacuum in a certain non-perturbative representation. The difference between the two cases is that classically there is no scale parameter, so all scales are arbitrary, in contrast to the quantum case. Also, in the classical case the vacuum consists of pairs of non-trivial center vortices so the net effect in SU(2) is a flux  $(-1)^2$ , which gives the classical vacuum the usual trivial appearance. However, the fact that this appearance can be understood from a center vortex point of view makes it perhaps more natural that this kind of vortex lattice also occurs in the quantum state. In the quantum case the necessary scale for the transverse size of each vortex is provided by quantum mechanics.

In this note we investigate a periodic vacuum (zero energy) solution of the classical massless SU(2) Yang Mills theory. The SU(2) field strength is thus assumed to vanish ( $A_\mu = A_\mu^a t_a$ ,  $t_a = \sigma_a/2$ ),

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = 0. \quad (1)$$

The trivial vacuum field  $A_\mu = 0$  of course satisfies this. Any other vacuum configuration must be of the form

$$A_\mu = A_\mu^a t_a = \frac{i}{g} \partial_\mu \Omega \Omega^\dagger. \quad (2)$$

Alternatively the unitary matrix  $\Omega$  can be expressed in terms of the field

$$\Omega = \text{P} \exp \left( -ig \int_R^x A_\mu dx^\mu \right), \quad (3)$$

where  $R$  is some arbitrary reference point. Our ansatz for the vacuum solution is based on the fields

$$A_\mu^3 \text{ with } \mu = 1, 2 \text{ and } W_\mu = \frac{1}{\sqrt{2}} (A_\mu^1 + iA_\mu^2), \quad W_2 = iW_1 \equiv iW, \quad W_3 = W_0 = 0. \quad (4)$$

We assume periodicity in the  $x_1 - x_2$  plane. The field  $W$  stabilizes the dynamics, since if it was not present, an instability would be generated, at least for sufficiently homogeneous fields [3]. Then the solution of  $F_{\mu\nu} = 0$  for the  $A^3$ -field satisfies the first order equations

$$(D_1 + iD_2)W = 0, \quad f_{12} = \partial_1 A_2^3 - \partial_2 A_1^3 = 2g^2 |W|^2. \quad D_i = \partial_i - igA_i^3 \quad (5)$$

Similar equations were used long time ago in the massive  $\text{SU}(2)$  case [4]. It is possible to show directly from (5) that the second order equations of motion are satisfied by use of

$$(D_1 - iD_2)(D_1 + iD_2)W = 0, \text{ and } [D_1, D_2] = -igf_{12} \quad (6)$$

which follows from the first equation (5). Hence

$$(D_1^2 + D_2^2 + 2gf_{12})W - 2g^2 |W|^2 W = 0, \quad (7)$$

which is precisely one of the equations of motion for our ansatz. The second equation is derived by simply differntiating the second equation in (5),

$$\partial_i f_{ij} = 2g^2 \epsilon_{ij} \partial_i |W|^2, \quad (8)$$

showing that the magnetic field is generated by a current from the complex vector fields.

The equations (5) can be reexpressed as

$$A_i^3 = \frac{\epsilon_{ij}}{g} \partial_j \log |W| + \frac{1}{g} \partial_i \chi, \quad (9)$$

where  $\chi$  is the phase of  $W$ , and  $W$  satisfies the Liouville equation,

$$-(\partial_1^2 + \partial_2^2) \ln |W| = 2g^2 |W|^2 - \epsilon_{ij} \partial_i \partial_j \chi. \quad (10)$$

These equations are non-perturbative. The magnetic field and the  $W$ -field are in a bootstrap situation: The field  $f_{12}$  is generated by a current arising from the charged  $W$ -field, and the latter appears in order to stabilize the magnetic field which would otherwise be unstable as discussed in [3].

To proceed we take for simplicity the periodic lattice to consist of quadratic cells  $\omega \times i\omega$  and a solution which has a non-trivial topology

$$W(z, \bar{z}) = \frac{\sqrt{2}}{g} \frac{|e_1| |\wp'(z)|}{|e_1|^2 + |\wp(z)|^2} e^{i\chi}, \quad \chi = \sum_i \arg(z - z_i), \quad z_i = \omega n + i\omega m \quad (11)$$

can be obtained. Here  $z_i$  are the first order zeros of  $W$  encircled by the phase  $\chi$ . Also,  $\wp$  is the doubly periodic Weierstrass function with periods  $2\omega$ ,  $2i\omega$ . The solution (11) has, however, periods  $\omega$ ,  $i\omega$  [5]. The constant  $e_1$  known in the theory of the Weierstrass function is carefully arranged<sup>1</sup> in Eq. (11) such that in one cell one only has one zero of  $W$ . In general a construction in terms of Weierstrass' function (or any other elliptic function) a la (11) leads to an even number of zeros. The flux would then be trivial<sup>2</sup> in SU(2).

Next we shall evaluate the Wilson loop taken along the sides of a fundamental lattice cell. The corners are placed at  $C_1 = -\omega/2 - i\omega/2$ ,  $C_2 = +\omega/2 - i\omega/2$ ,  $C_3 = +\omega/2 + i\omega/2$ , and  $C_4 = -\omega/2 + i\omega/2$ . We now make a transformation  $\Omega$  of the field  $A_\mu = A_\mu^a t_a$  along this loop,

$$\hat{A}_\mu = \Omega A_\mu \Omega^\dagger - \frac{i}{g} \Omega \partial_\mu \Omega^\dagger, \quad (12)$$

where

$$\Omega = \begin{Bmatrix} e^{i\chi/2} & 0 \\ 0 & e^{-i\chi/2} \end{Bmatrix} = e^{i\chi t_3} \quad (13)$$

This transformation accomplishes the “removal” of the gradient term in the field  $A_i^3$  in Eq. (9). The new fields are given by

$$\hat{A}_i^3 = \frac{\epsilon_{ij}}{g} \partial_j \log |W|, \quad W_1 = |W|, \quad W_2 = i|W|. \quad (14)$$

Therefore the phase  $\chi$  has been transformed away from both  $A_i^3$  and the  $W$ -fields. We note that for our ansatz

$$A_1^1 t_1 + A_1^2 t_2 = \begin{Bmatrix} 0 & \sqrt{2}W^* \\ \sqrt{2}W & 0 \end{Bmatrix} \quad \text{and} \quad A_2^1 t_1 + A_2^2 t_2 = \begin{Bmatrix} 0 & -i\sqrt{2}W^* \\ i\sqrt{2}W & 0 \end{Bmatrix}. \quad (15)$$

Therefore with  $\Omega$  given as in Eq. (13) we obtain

$$\Omega(A_1^1 t_1 + A_1^2 t_2) \Omega^\dagger = \sqrt{2}|W| t_1 \quad (16)$$

and similarly for the 2 components, so to sum up

$$\hat{A}_1 = \hat{A}_1^3 t_3 + \sqrt{2}|W| t_1, \quad \hat{A}_2 = \hat{A}_2^3 t_3 + \sqrt{2}|W| t_2. \quad (17)$$

It should be remarked that this transformation would be bad near the zeros of  $W$ , because the first term on the right hand side of Eq. (9) is singular at a zero, but this is exactly canceled by the gradient term in this equation, making the  $A^3$ -fields finite at the zeros. However, this problem does not occur along the contour  $\mathcal{C} = C_1 - C_2 - C_3 - C_4 - C_1$ , where the new field  $\hat{A}^3$  is perfectly finite.

Along the contour  $\mathcal{C}$  the fields are simplified in an essential manner. Thus, along  $\mathcal{C}$  the field  $|W|$  has maximum and no slope in the direction transverse to this contour. Therefore

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<sup>1</sup>More general versions of the solution (11) have been given in [6]

<sup>2</sup>This is because the flux  $\int f_{12} d^2x$  through one cell would then be an even number times  $2\pi$ . Since the field is associated with  $\sigma_3/2$  in SU(2), this gives 0,  $2\pi$  etc. for the encircling angle. With only one zero the corresponding angle is  $\pi, 3\pi$ , etc.

$\partial_2 \log |W| = 0$  along the lines  $C_1 - C_2$  and  $C_3 - C_4$ , and  $\partial_1 \log |W| = 0$  along the lines  $C_2 - C_3$  and  $C_4 - C_1$ . Therefore it follows from Eqs. (14) that along the  $C_1 - C_2$  and  $C_3 - C_4$  lines

$$\hat{A}_1^3 = 0 \text{ on } C_1 - C_2 \text{ and } C_3 - C_4. \quad (18)$$

Similarly

$$\hat{A}_2^3 = 0 \text{ on } C_2 - C_3 \text{ and } C_4 - C_1. \quad (19)$$

It is therefore a consequence that the  $\hat{A}$  field only has contributions from  $|W|$ , as is seen from Eq. (17),

$$\hat{A}_1 = \sqrt{2}|W| t_1 \text{ on } C_1 - C_2 \text{ and } C_3 - C_4, \quad (20)$$

and

$$\hat{A}_2 = \sqrt{2}|W| t_2 \text{ on } C_2 - C_3 \text{ and } C_4 - C_1, \quad (21)$$

This is the important simplification which allows us to compute the Wilson loop around the boundary of the fundamental cell  $\mathcal{C} = C_1 - C_2 - C_3 - C_4 - C_1$ .

We have

$$W(\mathcal{C}) = \text{tr P} \exp \left( ig \int_{\mathcal{C}} A_\mu dx_\mu \right) = \text{tr P} \left[ \Omega_{\text{initial}}^\dagger \exp \left( ig \int_{\mathcal{C}} \hat{A}_\mu dx_\mu \right) \Omega_{\text{final}} \right]. \quad (22)$$

Now  $\Omega_{\text{final}}$  differs from  $\Omega_{\text{initial}}$  by the center element (-1). Therefore

$$W(\mathcal{C}) = (-1)_{A^3} \text{tr P} \exp \left( ig \int_{\mathcal{C}} \hat{A}_\mu dx_\mu \right). \quad (23)$$

We have written the (-1) in this special way in order to remind us that this center contribution comes from the original field  $A_3$ .

Next point is that the integral along the different paths is the same, due to the fact that the function  $|W|$  is symmetric in  $x_1$  and  $x_2$ , so we have

$$\begin{aligned} g \int_{c_1-c_2} \hat{A}_1 dx_1 &= g \int_{c_2-c_3} \hat{A}_2 dx_2 \equiv 2I \\ g \int_{c_3-c_4} \hat{A}_1 dx_1 &= g \int_{c_4-c_5} \hat{A}_2 dx_2 \equiv -2I, \quad I = \frac{g}{\sqrt{2}} \int_{C_1-C_2} |W(x_1 - i\omega/2)| dx_1. \end{aligned} \quad (24)$$

This is easily seen because the function  $|W|$  only depends on  $x_1 + ix_2$ , and therefore the integrations along  $C_1 - C_2$  and  $C_2 - C_3$  etc. produce the same results. The minus signs simply arise from the inversion of the paths of integration, taking into account the periodicity of  $|W|$ .

Collecting these results we obtain

$$W(\mathcal{C}) = (-1)_{A^3} \text{tr} \left[ e^{i\sigma_1 I} e^{i\sigma_2 I} e^{-i\sigma_1 I} e^{-i\sigma_2 I} \right] \quad (25)$$

By use of the relation  $e^{i\sigma_1 I} = \cos I + i\sigma_1 \sin I$ , etc., we easily obtain

$$W(\mathcal{C}) = (-1)_{A^3} (1 - 2 \sin^4(I))_W, \quad (26)$$

where we used the commutator

$$e^{i\sigma_2 I} e^{-i\sigma_1 I} - e^{-i\sigma_1 I} e^{i\sigma_2 I} = [\sigma_2, \sigma_1] \sin^2[I] = -2i \sin^2[I] \sigma_3. \quad (27)$$

The index  $W$  on  $1 - 2\sin^4(I)$  is there to remind us that this contribution comes from the  $W$ -field.

The integral  $I$  is explicitly given by

$$I = \int_{-\omega/2}^{\omega/2} dx_1 \frac{e_1 |\wp'(x_1 - i\omega/2)|}{e_1^2 + |\wp(x_1 - i\omega/2)|^2}. \quad (28)$$

Scaling the complex variable by  $\omega$  this integral can be written

$$I = \int_{-1/2}^{1/2} du_1 \sqrt{c} \frac{|\wp'(u_1 - i/2)|}{c + |\wp(u_1 - i/2)|^2}, \quad c = (e_1 \omega^2)^2 = \frac{g_2}{4} \omega^4 = 15 \sum_{mn} \frac{1}{(2n + 2im)^4}, \quad (29)$$

where  $m, n$  are different from  $(0,0)$ . The Weierstrass function above is periodic in the cell  $2 + i2$ . The constant  $c$  is therefore independent of  $\omega$ , so  $I$  is independent of  $\omega$ , and we only need to evaluate  $I$  once. I have not been able to compute this integral analytically, but a high precision numerical integration gives<sup>3</sup>

$$I = \frac{\pi}{2}. \quad (30)$$

Inserting this in (26) we obtain the result

$$W(\mathcal{C}) = (-1)_{A^3} (-1)_W \quad (31)$$

The result is therefore that the Wilson loop gets a center vortex contribution from the  $A^3$ -field and another such contribution from the  $W$ -field. Since  $(-1)^2 = +1$  the contribution from this pair of center vortices is the trivial unit element, corresponding to the natural expectation that the *classical* vacuum does not carry a magnetic flux, in accordance with the possibility of gauge transforming all fields  $A_\mu$  to zero. A similar conclusion can be obtained in a simpler manner by using that when the field strength  $F_{\mu\nu}$  vanishes, the Wilson loop can be shown to be independent of the loop [7] when there are no singularities. Therefore the contour can be contracted to a small circle around the zero, and again we get two contributions which cancel.

In the case of the electroweak  $SU(2) \times U(1)$  theory there is a similar magnetic vortex lattice representation of the vacuum in the symmetric phase. This can be verified using the same methods as above. We also mention that our approach may have applications for Chern Simons dynamics.

The conclusion is thus that the classical  $SU(2)$  vacuum is a sea of pairs of non-trivial center vortices which combine to triviality. Perturbations actually converts the magnetic flux so as to become physical. For example, adding a mass term the center vortex state appears in a non-trivial [4] manner. Here the Liouville equation (10) is replaced by

$$-(\partial_1^2 + \partial_2^2) \ln |W| = m^2 + 2g^2 |W|^2 - \epsilon_{ij} \partial_i \partial_j \chi, \quad (32)$$

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<sup>3</sup>By use of WolframAlpha on a smartphone I obtained  $I - \pi/2 = 2.157 \times 10^{-6}$ , where  $g_2 = 11.8171$

where  $m$  is a mass added to the Lagrangian by a term  $-m^2 W_\mu W_\nu^\dagger$ . The resulting vortices now have a scale set by the mass  $m$ . A similar result is true in the electroweak  $SU(2) \times U(1)$  theory, where the Higgs field breaks the triviality [5]. This is directly related to a phase transition from the massless to the massive case.

For the case of the quantum fluctuations the situation was discussed by Ambjørn and the author [8], where we considered the Savvidy magnetic field  $H_S$  [9] as a background field. The Liouville equation is then replaced by

$$-(\partial_1^2 + \partial_2^2) \ln |W| = gH_S - \epsilon_{ij} \partial_i \partial_j \chi, \quad (33)$$

which is similar to Eq. (32) with the mass replaced by the Savvidy field  $H_S$  and also assuming that  $W$  is much smaller than  $H_S$ . This assumption makes it possible to ignore the  $2g^2|W|^2$  term which in general would occur on the right hand side like in Eq. (32). The solution of Eq. (33) is different from Eq. (11). An example of a solution is [8]

$$W(z) = \text{const. } e^{-gH_S x^2/2} \theta(z), \quad (34)$$

where  $\theta$  is a theta function. For a detailed discussion we refer to ref. [8]. The integral which replaces  $I$  is now much smaller than  $\pi/2$  because  $W$  is small, so in the resulting Wilson loop the  $(-1)_{A^3}$  contribution cannot be overwhelmed by the  $1 - 2\sin^4 I$  factor in Eq. (26). We therefore get a non trivial  $SU(2)$  center vortex.

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